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HOW MANY HERONIAN TRIANGLES HAVE A GIVEN LONGEST SIDE? A COMPLETE COUNT UP TO 2000 AND THE RIGHT TRIANGLES HIDDEN INSIDE THEM

Abstract: A Heronian triangle has integer sides and an integer area. Sequence A054875 of the On-Line Encyclopedia of Integer Sequences (OEIS) gives the number $a(n)$ of Heronian triangles whose longest side is exactly n , but its entry has no formula and no references. I wrote down three questions (for which n is $a(n) = 0$, what share of the triangles is right-angled, and what share is glued from two integer right triangles along an altitude) and made a forecast: about 300 triangles with longest side at most 100. All 668,167,500 integer triangles with longest side at most 2000 were then checked with Heron's formula, and a second enumeration by a different parametrisation gave the same list. There are 40,764 Heronian triangles with longest side at most 2000, and the counts agree term by term with the OEIS tables. The forecast was close: there are 321 such triangles up to 100. The total grows faster than the side itself (the average of $a(n)$ is 4.6 for $50 < n \leq 100$ and 26.7 for $1000 < n \leq 2000$), but $a(n) = 0$ for 333 values of $n \leq 2000$. None of these n has a prime factor of the form $4k + 1$, and the paper proves that $a(n) = 0$ whenever n is a power of a prime $p \equiv 3 \pmod{4}$ (for primes this was recorded in the OEIS as an observation) or $n = 2p$ for such a prime $p \geq 7$; these two families give 254 of the 333 zeros. Right triangles make up only 4.9% of the 40,764, and their share falls across the range. Of all triangles, 32.6% are glued from two integer right triangles, 30.6% are the difference of two such triangles, and 32.0% have no integer altitude at all; for $1000 < n \leq 2000$ this last group is already the largest. Finally, as also follows from earlier work [3, 4], every altitude of a Heronian triangle is a fraction whose denominator has only prime factors of the form $4k + 1$.

Keywords: Heronian triangle; Pythagorean triple; integer sequence; OEIS A054875; Diophantine equation; enumeration

Introduction

By Heron's formula, a triangle with sides a, b, c has area $S = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$. Usually the square root leaves an irrational number, but sometimes it does not: the triangle 3, 4, 5 has area 6 and the triangle 13, 14, 15 has area 84. Triangles with integer sides and integer area are called Heronian. Carlson [1] showed that every Heronian triangle is obtained by joining two Pythagorean triangles along a common side, possibly followed by dividing all sides by a common factor. Kurz [2] described algorithms that list all Heronian triangles up to a given longest side and generated the complete list up to 600,000. He found no estimates in the literature of how many such triangles there are, and showed that there are $O(n^{1+\varepsilon})$ Heronian triangles with longest side n , for every $\varepsilon > 0$. Stănică and co-authors [3] bounded the number of Heronian triangles with two fixed sides, counted those with a fixed

height and studied when a Heronian triangle can be cut into two smaller ones, and Yiu [4] studied the Heronian triangles that cannot be cut into two integer right triangles at all.

The OEIS entry A054875 [5] counts $a(n)$, the number of Heronian triangles (up to congruence) whose longest side is exactly n . Its first values are 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 2, 0, 3. The entry contains a table of the first 10,000 terms and two short programs, but no formula and no references to the literature. The neighbouring entries [6] count the same triangles by the second-longest side, count only the primitive ones, or list the triangles themselves. One of them, A096467, records the observation that a prime of the form $4k + 3$ never seems to be the longest side of a Heronian triangle and refers to an article by Cheney (1929), which we could not access.

In simple words, we are looking for "perfect" integer triangles: triangles in which both the sides and the area come out as whole numbers, without any fractions or roots. Sequence A054875 counts how many of them can be built when the longest side is fixed at a number n . The fact that caught my attention is this. If n is really the longest side, the altitude dropped onto it always falls inside the triangle. Because the area is an integer, this altitude cuts the triangle into two right triangles whose sides are rational numbers, that is, into two rational Pythagorean triangles. So every triangle in our count is two right triangles glued together so that their bases meet exactly and add up to n . We seem to look only at integer sides and areas, but inside, the problem quietly assembles a hidden puzzle out of Pythagorean triangles.

Knowing the first OEIS values, which my mentor sent me on 24 September, I sent a list of all Heronian triangles with longest side at most 10, with a proof that there are exactly four (4 and 8 October 2026): 3, 4, 5; 5, 5, 6; 5, 5, 8 and 6, 8, 10. Two of them show the gluing directly: 5, 5, 6 and 5, 5, 8 are two copies of the triangle 3, 4, 5 joined along the leg 4 and along the leg 3. The proof used two steps that are used again below: the perimeter of a Heronian triangle is even (Lemma 1, a classical fact), and the substitution $a = u + v$, $b = u + w$, $c = v + w$, known as the Ravi substitution and also used by Yiu [4, p. 5], turns Heron's formula into $S^2 = uvw(u + v + w)$.

On 6 October 2026, following examples my mentor suggested the day before, I wrote down three questions: (Q1) for which n do such triangles exist, and for which n is $a(n) = 0$; (Q2) what share of the triangles is right-angled when n is large; (Q3) what share is obtained by gluing two right triangles along an altitude, and which kinds of gluing occur most often. My mentor then set the range $n \leq 2000$. On 9 October 2026 I made a forecast. By then I had seen the OEIS page, which lists $a(n)$ for $n \leq 83$ (these values add up to 229), and my own list up to 10. My forecast, in full, was: "My guess is about 300, because if n increases, there are more ways to choose the other two sides, which also makes the probability of 'Heronian' triangles larger." This paper reports the complete count up to 2000, compares it with the OEIS tables and with my forecast, and answers the three questions, partly by computation and partly by proof.

Materials and methods

Definitions. A triangle is written with its sides in order $a \leq b \leq c$, so congruent triangles are counted once. It is Heronian if a , b , c and the area S are positive integers. It is primitive if $\gcd(a, b, c) = 1$. Let $a(n)$ be the number of Heronian triangles with $c = n$, and $H(N) = a(1) + a(2) + \dots + a(N)$ the number with longest side at most N .

Enumeration, method A. Every triple of integers with $a \leq b \leq c \leq 2000$ and $a + b > c$ was checked; there are 668,167,500 such triples. A triple was kept when

$$16S^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c)$$

is a perfect square and S is an integer. This is the direct algorithm that Kurz describes as the fastest one for small sides [2]. The square test was done with exact integer arithmetic.

Enumeration, method B (the Ravi substitution). By Lemma 1 below, $s = (a + b + c)/2$ is an integer, so $u = s - c$, $v = s - b$, $w = s - a$ are positive integers with $1 \leq u \leq v \leq w$, and $a = u + v$, $b = u + w$, $c = v + w$. Then $S^2 = uvw(u + v + w)$, and the method keeps the triples (u, v, w) with $v + w \leq 2000$ for which this number is a perfect square. Method B never looks at triangles with an odd perimeter, so its agreement with method A also checks Lemma 1 on every triangle.

Both enumerations are part of one Python program (with NumPy) and use the same perfect-square test, so method B is a second enumeration by a different parametrisation rather than fully independent code. They took 16 s and 10 s on a laptop and produced the same list of triangles, one by one.

Checks against the OEIS. The counts were compared with the tables published in the OEIS: $a(n)$ with A054875 [5]; the counts by the second-longest side with A054876, which our list determines completely for $n \leq 1000$ because $c < a + b \leq 2b$; the primitive counts with A239246; the first 10,000 triangles of the ordered list (sides and area) with A055592, A055593, A055594 and A055595; the set of longest sides of primitive triangles with A096467; and the positions of the records of $a(n)$ with A322105 [6].

Classes for Q2 and Q3. For each side X of a triangle, the altitude onto X has length $h = 2S/X$. Its foot lies strictly inside X exactly when both angles at the ends of X are acute. Each triangle was put into one of four classes:

- right: $a^2 + b^2 = c^2$;
- glued: not right, and some altitude whose foot lies inside its side is an integer, so the triangle consists of two integer right triangles placed side by side (Lemma 2 below), as 13, 14, 15 = (5, 12, 13) + (9, 12, 15);
- difference: not right, not glued, but some altitude is an integer and its foot lies outside its side, so the triangle is one Pythagorean triangle cut out of a larger one, as 4, 13, 15 = (9, 12, 15) – (5, 12, 13);
- indecomposable: not right and no altitude is an integer, as 5, 29, 30.

Together, "glued" and "difference" form what Yiu calls decomposable triangles [4]. Glued triangles were also sorted by the kind of gluing: two copies of the same right triangle (the triangle is then isosceles) or two different right triangles; along the longest side or only along a shorter side; and in how many different ways. The classification was compared with the OEIS lists of areas of primitive indecomposable triangles A227166 and A239978 [7] (A239978 also includes the primitive right triangles).

Results

The count and its agreement with the OEIS. There are 40,764 Heronian triangles with longest side at most 2000, and 16,719 of them (41.0%) are primitive. The two enumerations gave identical lists, and no triangle with an odd perimeter appeared. Every comparison with the OEIS agreed (Table 1). The largest value of $a(n)$ in the range is $a(1950) = 210$.

Table 1. Comparison of the computed list with the OEIS tables

What was compared	Range	Result
$a(n)$, A054875	n from 1 to 2000	all 2000 values equal
primitive triangles by longest side, A239246	n from 1 to 2000	all 2000 values equal
triangles by second-longest side, A054876	n from 1 to 1000	all 1000 values equal

ordered list of sides and areas, A055592 to A055595	first 10,000 triangles	identical
longest sides of primitive triangles, A096467	values up to 2000	identical sets
positions of records of $a(n)$, A322105	$n \leq 2000$	the same 27 positions
areas of primitive indecomposable triangles, A227166 and A239978 (A239978 also includes the primitive right triangles)	all 48 and 51 listed terms	identical
method A against method B	longest side ≤ 2000	identical lists of 40,764 triangles

The forecast and the growth of the count. My forecast of about 300 triangles with longest side at most 100 was close: $H(100) = 321$, so it was 21 triangles (6.5%) too low. The first reason I gave was right on average. The mean of $a(n)$ over the second half of the range rises from 4.56 for $50 < n \leq 100$ to 26.67 for $1000 < n \leq 2000$ (Figure 1, Table 2), so $H(N)$ grows faster than N : $H(N)/N$ rises from 3.21 at $N = 100$ to 20.38 at $N = 2000$. It grows much slower than N^2 , however. On a logarithmic scale the slope of $H(N)$ is 1.71 between $N = 100$ and 200, 1.57 between 500 and 1000, and 1.53 between 1000 and 2000 (Figure 2), and in the OEIS table [5] it is 1.51 between 2000 and 5000 and 1.47 between 5000 and 10,000. In the same table, for $1000 \leq N \leq 10,000$ the ratio $H(N)/N^{1.5}$ stays between 0.443 and 0.462. This is only an observation on a finite range, not a law; it is far below the bound $H(N) = O(N^{2+\epsilon})$ that follows from Kurz's estimate [2]. The second reason in my forecast, a larger probability, was wrong: the chance that a random integer triangle is Heronian falls quickly. Heronian triangles are 0.37% of all integer triangles with longest side at most 100 and 0.0061% of those with longest side at most 2000.

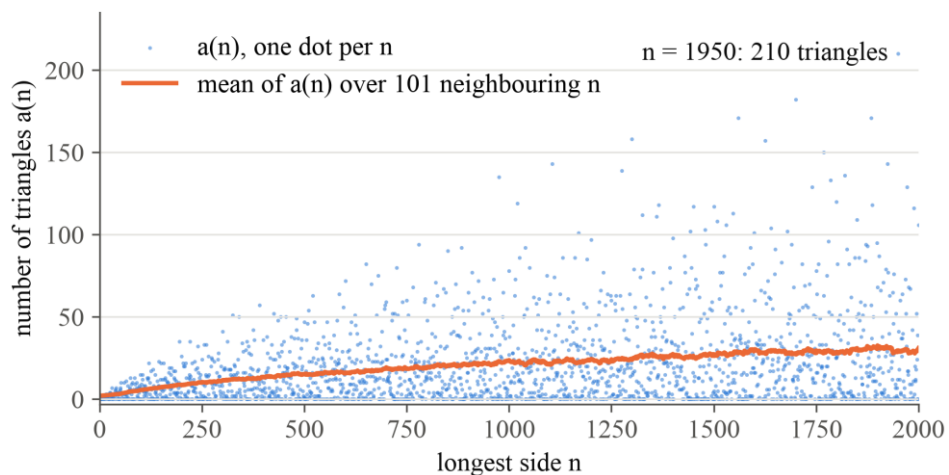


Figure 1. The number $a(n)$ of Heronian triangles with longest side n , for $n \leq 2000$. The dots on the horizontal axis are the 333 values of n with $a(n) = 0$.

Table 2. Growth of the count of Heronian triangles

N	H(N)	H(N)/N	mean of a(n) for $N/2 < n \leq N$	$H(N)/N^{1.5}$
100	321	3.21	4.56	0.321
200	1,050	5.25	7.29	0.371
500	4,740	9.48	12.86	0.424
1000	14,090	14.09	18.70	0.446
2000	40,764	20.38	26.67	0.456
5000*	162,182	32.44	41.93	0.459
10,000*	449,510	44.95	57.47	0.450

* from the OEIS table of A054875 [5]; these two rows were not recomputed.

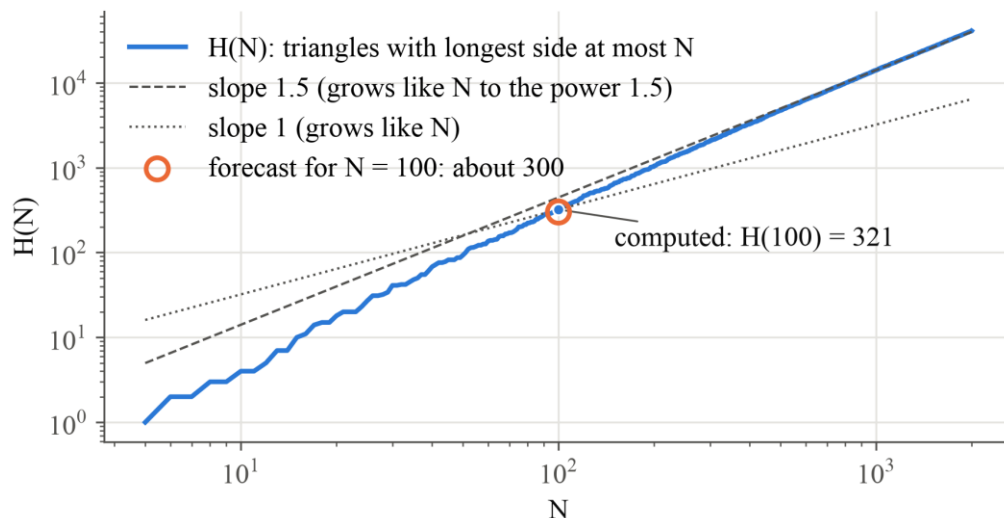


Figure 2. $H(N)$, the number of Heronian triangles with longest side at most $N \leq 2000$, on logarithmic axes, with reference lines of slope 1 and 1.5 and the forecast for $N = 100$.

The individual values, however, do not grow steadily. Figure 1 shows that $a(n) = 0$ keeps happening for large n , which is the subject of the first question.

Q1: for which n is $a(n) = 0$. For $n \leq 2000$ there are 333 zeros. The following facts explain most of them. Lemma 1 is classical and was also the first step of the list I sent on 4 October; Lemma 2 is my observation about the altitude written as a formula.

Lemma 1. A Heronian triangle has an even perimeter. *Proof.* Consider $16S^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c)$. Each of the last three factors differs from $a + b + c$ by $2a$, $2b$ or $2c$, so all four factors have the same parity. The left-hand side is even, so the factors cannot all be odd; hence all four are even and $a + b + c$ is even. (For primitive triangles this also follows from the fact that exactly one side is even [1, §4].)

Lemma 2. Let $h = 2S/c$ be the altitude onto the side c , and let $x = (b^2 + c^2 - a^2)/(2c)$ be the piece of c next to the side b . Then $h^2 + x^2 = b^2$, and after multiplying by $(2c)^2$

$$(4S)^2 + (b^2 + c^2 - a^2)^2 = (2bc)^2. \quad (1)$$

So the right triangle with legs h , x and hypotenuse b becomes a triangle with integer sides when it is enlarged $2c$ times. If c is the longest side, both angles at its ends are acute, so the foot of h lies inside c and the triangle is the union of two right triangles with rational sides, (h, x, b) and $(h, c - x, a)$. If h is an integer, then $x^2 = b^2 - h^2$ is an integer and x is rational, so x is an integer too, and the two pieces are Pythagorean triangles. The same argument applies to any side whose altitude foot lies inside it.

Lemma 3. (i) Every prime $p \equiv 1 \pmod{4}$ is a sum of two squares, $p = u^2 + v^2$ [3, Lemma 2; 8, Theorem 13; 9, Theorem 5.7.1], so it is the hypotenuse of the right triangle $(u^2 - v^2, 2uv, p)$. (ii) Let $x^2 + y^2 = z^2$ with integers x, y and $z \neq 0$, and let p be a prime with $p = 2$ or $p \equiv 3 \pmod{4}$. If p^e is the largest power of p dividing z , then p^e divides both x and y . This is clear if x or y is 0, and signs do not matter, so let x, y, z be positive. For $p \equiv 3 \pmod{4}$ the claim holds because p dividing $x^2 + y^2$ forces p to divide x and y [8, proof of Theorem 12; 9, Lemma 5.7.4]; dividing everything by p and repeating gives the full power. For $p = 2$, write $(x, y, z) = g(x_0, y_0, z_0)$ with a primitive triple; z_0 is odd [8, Theorem 3], so the power of 2 in z is the power of 2 in g , which divides x and y .

Proposition 1. If n has a prime factor $p \equiv 1 \pmod{4}$, then $a(n) \geq 1$. *Proof.* By Lemma 3(i), the right triangle $(u^2 - v^2, 2uv, p)$ multiplied by n/p has hypotenuse n , which is its longest side.

Proposition 2. If $n = p^k$, where p is a prime with $p \equiv 3 \pmod{4}$ and $k \geq 1$, then $a(n) = 0$. *Proof.* Suppose $a \leq b \leq n$ is a Heronian triangle with longest side n . In (1) with $c = n$ the number $2bn$ is divisible by p^k , so by Lemma 3(ii) p^k divides $b^2 + n^2 - a^2$, and therefore p^k divides $(b - a)(b + a)$. If p divided both $b - a$ and $b + a$, it would divide a and b , and the triangle $a/p, b/p, p^{k-1}$ would again be Heronian (its area is S/p^2 , an integer [1, Lemma 1]). Otherwise p^k divides one of the two factors. If p^k divides $b - a$, then $a = b$ because $0 \leq b - a < p^k$, and the perimeter $2a + p^k$ is odd, which contradicts Lemma 1. If p^k divides $b + a$, then $a + b = 2p^k$ because $p^k < a + b \leq 2p^k$, so $a = b = n$ and the triangle is equilateral, with the irrational area $(\sqrt{3}/4)n^2$. Thus a Heronian triangle with longest side p^k would give one with longest side p^{k-1} , then p^{k-2} , and finally one with longest side 1, which does not exist.

Proposition 3. If $n = 2p$, where $p \equiv 3 \pmod{4}$ is a prime and $p \geq 7$, then $a(n) = 0$. *Proof.* Let $a \leq b \leq 2p$ be a Heronian triangle. As in Proposition 2, p divides $(b - a)(b + a)$. Because $0 \leq b - a < 2p$ and $2p < a + b \leq 4p$, there are four cases: $b - a = p$, $a + b = 3p$, $a + b = 4p$, or $a = b$. In the first two cases the perimeter $a + b + 2p$ is odd, which contradicts Lemma 1; the third case is the equilateral triangle. In the last case the triangle $a, a, 2p$ has the altitude $h = S/p$ onto its base, $h^2 = a^2 - p^2$ is an integer and h is rational, so h is an integer and $(a - h)(a + h) = p^2$. Since $a - h < a + h$, this gives $a - h = 1$ and $a = (p^2 + 1)/2$. But $a \leq 2p$, so $p^2 - 4p + 1 \leq 0$ and $p \leq 3$. For $p = 3$ we get $a = 5$, the triangle 5, 5, 6 from my list; for $p \geq 7$ there is no triangle.

Table 3 sorts all $n \leq 2000$ by these results. Every one of the 1187 values of n with a prime factor $4k + 1$ has $a(n) \geq 1$, as Proposition 1 says. All 168 prime powers p^k with $p \equiv 3 \pmod{4}$, among them all 155 primes $p \equiv 3 \pmod{4}$ up to 2000, are zeros, and so are all 86 numbers $2p$ with $p \geq 7$. For primes, Proposition 2 proves the observation recorded in A096467 [6]. In the OEIS table up to 10,000 [5] the two families contain 642 and 338 numbers, and all of them are zeros as well. The set of zeros is therefore infinite (it contains 3, 9, 27, 81, ...), so $a(n)$ does not grow for every n , even though it grows on average.

Table 3. The values $n \leq 2000$ sorted by the form of n

Form of n	Values of n	Zeros	Reason
has a prime factor $4k + 1$	1187	0	Proposition 1
p^k , p prime, $p \equiv 3 \pmod{4}$	168	168	Proposition 2
$2p$, p prime, $p \equiv 3 \pmod{4}$, $p \geq 7$	86	86	Proposition 3
1, 2, 4	3	3	the list for $c \leq 10$
all other n without a prime factor $4k + 1$	556	76	not explained
total	2000	333	

The absence of a prime factor $4k + 1$ is necessary for a zero but not sufficient: 480 such $n \leq 2000$ do have triangles. The smallest are 6 and 8, the bases of the triangles 5, 5, 6 and 5, 5, 8 from my list. The first odd one is 21, which is not the hypotenuse of any right triangle but is the sum of two legs: $21 = 6 + 15$ gives 10, 17, $21 = (6, 8, 10) + (15, 8, 17)$, and $21 = 5 + 16$ gives 13, 20, $21 = (5, 12, 13) + (16, 12, 20)$. The 76 remaining zeros, the smallest of which are 33, 57, 76, 98, 99 and 124, are not covered by Propositions 1, 2 and 3.

Q2: the share of right triangles. Every Pythagorean triangle is Heronian, but there are few of them: 52 of the 321 triangles up to 100 (16.2%), 881 of 14,090 up to 1000 (6.3%) and 1,981 of 40,764 up to 2000 (4.9%) (Table 4). In consecutive windows of the longest side the share falls from 14.0% for $50 < n \leq 100$ to 4.1% for $1000 < n \leq 2000$ (Table 5). From $N = 1000$ to $N = 2000$ the number of right triangles grows 2.25 times while the number of all Heronian triangles grows 2.89 times, so on this range about 95% of Heronian triangles are not right-angled, and the share of right triangles keeps falling.

Q3: glued triangles and the other classes. Figure 3 shows the two situations that Lemma 2 allows. In 13, 14, 15 the altitude onto the side 14 is 12, an integer, and the triangle is glued from (5, 12, 13) and (9, 12, 15). In 15, 34, 35 the altitude onto the longest side is $72/5$, so the two pieces exist but have rational sides; the same triangle enlarged 5 times, 75, 170, 175, is glued from (21, 72, 75) and (154, 72, 170).

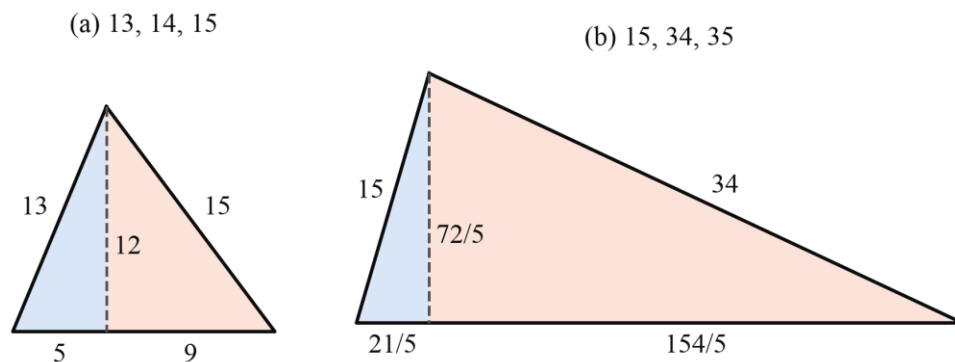


Figure 3. The altitude idea. (a) In 13, 14, 15 the altitude 12 onto the side 14 is an integer: the triangle is glued from (5, 12, 13) and (9, 12, 15). (b) In 15, 34, 35 the altitude onto the longest side is $72/5$; after multiplying all lengths by 5 the triangle 75, 170, 175 is glued from (21, 72, 75) and (154, 72, 170).

Table 4 gives the four classes for all triangles up to N . Up to 2000, 13,270 triangles (32.6%) are glued, 12,484 (30.6%) are differences of two Pythagorean triangles, 13,029 (32.0%) are indecomposable and 1,981 (4.9%) are right. The smallest indecomposable triangle is 5, 29, 30 with area 72 and the smallest acute indecomposable one is 15, 34, 35 with area 252, as in Yiu's talk [4]; the difference triangle with the smallest longest side is 4, 13, 15 with area 24. The shares change with n (Table 5, Figure 4): the glued share falls from 43.9% for $50 < n \leq 100$ to 31.3% for $1000 < n \leq 2000$, the difference share stays between 30% and 32%, and the indecomposable share rises from 11.4% to 34.2%, so for $1000 < n \leq 2000$ indecomposable triangles are the largest class.

Table 4. Classes of Heronian triangles with longest side at most N

N	All	Right	Glued	Difference	Indecomposable
100	321	52 (16.2%)	145 (45.2%)	92 (28.7%)	32 (10.0%)
500	4,740	386 (8.1%)	1,764 (37.2%)	1,492 (31.5%)	1,098 (23.2%)
1000	14,090	881 (6.3%)	4,908 (34.8%)	4,392 (31.2%)	3,909 (27.7%)
2000	40,764	1,981 (4.9%)	13,270 (32.6%)	12,484 (30.6%)	13,029 (32.0%)

Table 5. Classes of Heronian triangles in consecutive windows of the longest side n

Window of n	All	Right	Glued	Difference	Indecomposable
51 to 100	228	32 (14.0%)	100 (43.9%)	70 (30.7%)	26 (11.4%)
101 to 250	1,205	113 (9.4%)	468 (38.8%)	385 (32.0%)	239 (19.8%)
251 to 500	3,214	221 (6.9%)	1,151 (35.8%)	1,015 (31.6%)	827 (25.7%)
501 to 1000	9,350	495 (5.3%)	3,144 (33.6%)	2,900 (31.0%)	2,811 (30.1%)
1001 to 2000	26,674	1,100 (4.1%)	8,362 (31.3%)	8,092 (30.3%)	9,120 (34.2%)

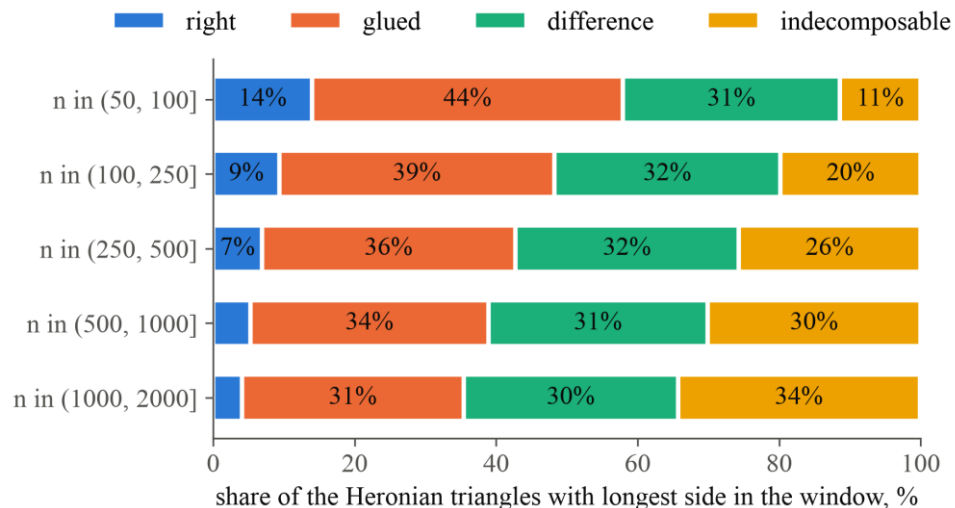


Figure 4. Classes of Heronian triangles in the five consecutive windows of Table 5 (n from 51 to 2000; below 51 there are only 93 triangles). Labels below 6% are omitted; all counts and shares are in Table 5.

The kinds of gluing are uneven. Of the 13,270 glued triangles, 10,412 (78.5%) are made of two different right triangles and 2,858 (21.5%) of two copies of the same right triangle. The 2,858 mirror gluings are exactly the 2,858 isosceles Heronian triangles, in agreement with the result of Stănică and co-authors that every isosceles Heronian triangle can be cut along the altitude onto its base [3, Proposition 16]. In 1,824 of them the common altitude is the longer leg of the right triangle, as in 5, 5, 6, and in 1,034 it is the shorter leg, as in 5, 5, 8. Gluing along the longest side, the picture from my paragraph in the introduction, gives 8,300 of the glued triangles (62.5%); the other 4,970 can be glued only along a shorter side, like 13, 14, 15. Only 349 triangles (2.6% of the glued ones) can be glued in two or three different ways, for example 25, 25, 30, and none of them is primitive, as Yiu proved for all primitive triangles [4, Theorem 1].

My observation about the altitude onto the longest side can now be measured. That altitude is rational for all 40,764 triangles, as $h = 2S/c$ is a quotient of integers, but it is an integer for only 8,404 of them (20.6%): the 8,300 triangles glued along the longest side and 104 right triangles that can be cut along the hypotenuse. For the other 79.4% the hidden pieces have rational, not integer, sides. Their denominators are not arbitrary.

Proposition 4. For every Heronian triangle and every side c , the altitude $h = 2S/c$ onto that side and the two numbers $x = (b^2 + c^2 - a^2)/(2c)$ and $c - x = (a^2 + c^2 - b^2)/(2c)$, which give the pieces of c cut off by the foot of the altitude, are fractions whose denominators, in lowest terms, have only prime factors of the form $4k + 1$. For the altitude this also follows from [3, proof of Theorem 4] and [4, Proposition 3 and Theorem 9]; we give a short direct proof. *Proof.* Let $p = 2$ or $p \equiv 3 \pmod{4}$ be a prime and let p^e be the largest power of p dividing c . By (1) and Lemma 3(ii), the largest power of p dividing $2bc$ divides both $4S$ and $b^2 + c^2 - a^2$. For odd p this gives $p^e \mid S$ and $p^e \mid (b^2 + c^2 - a^2)$; for $p = 2$ it gives $2^{e+1} \mid 4S$ and $2^{e+1} \mid (b^2 + c^2 - a^2)$. In both cases p does not divide the denominator of $2S/c$ or of $(b^2 + c^2 - a^2)/(2c)$. Exchanging a and b gives the same for $c - x$.

So in Carlson's description [1], when the foot of an altitude lies inside its side, the factor by which the triangle must be enlarged to become glued along that side is always a number whose prime factors all have the form $4k + 1$, such as 5, 13, 17, 25 or 29. In our list the denominators of the altitude onto the longest side are exactly of this kind: 1 for 8,404 triangles, 5 for 4,384, 13 for 2,891, 17 for 2,387 and 29 for 1,486, and the program found no exception among the 122,292 pairs of a triangle and one of its sides, for the altitudes or for the pieces. For the 13,029 indecomposable triangles, the smallest enlargement that makes the triangle glued is 5 in 3,475 cases (26.7%), as for 5, 29, 30 and 15, 34, 35.

Discussion

The three questions have clear answers on the range studied. For Q1, a zero of $a(n)$ is impossible when n has a prime factor $4k + 1$ (Proposition 1), and certain when n is a power of a prime $4k + 3$ or twice such a prime above 3 (Propositions 2 and 3). The proofs rest on identity (1), which is the "hidden puzzle" from the introduction written with integers, together with Lemma 3, the scaling lemma [1, Lemma 1] and the parity lemma. Parity removes one of the three cases in Proposition 2 and two of the four in Proposition 3. Seventy-six zeros up to 2000 remain unexplained, and the first of them, $33 = 3 \cdot 11$, is a natural next problem.

For Q2, right triangles are a small and shrinking part of all Heronian triangles on this range. For Q3, the picture of two right triangles glued along an altitude is always true with rational pieces, but with integer pieces it holds for a third of the triangles and becomes less common as n grows, while triangles with no integer altitude at all become the largest class. The denominators that appear are always products of primes $4k + 1$ (Proposition 4), the same primes that divide the hypotenuses of primitive Pythagorean triangles [3, Lemma 2], so the hidden pieces are still close relatives of Pythagorean triangles even when they are not integers.

My forecast was close in size. Its first reason, more ways to choose the other two sides, was right on average but not for every n : the count $H(N)$ grows faster than N , yet $a(n) = 0$ for infinitely many n . Its second reason, a larger probability, was wrong, because the chance that a random integer triangle is Heronian falls. Kurz [2] found no estimates in the literature and showed that $a(n) = O(n^{1+\varepsilon})$, so $H(N) = O(N^{2+\varepsilon})$; the exponent near 1.5 observed here is far below this bound, and we do not have a proof of it. We did not find Propositions 2 and 3 stated in the works we read [1, 2, 3, 4, 8, 9]. The nearest known results we found are in [3]: Corollary 5 (no Heronian triangle has two sides a and b such that every prime factor of ab is $\equiv 3 \pmod{4}$) and

Theorem 8 (no Heronian triangle has two odd prime sides $p \leq q$ with $q \equiv 3 \pmod{4}$). Neither implies Propositions 2 and 3, because the other sides of a triangle with longest side p^k or $2p$ need not be prime and may be even or have prime factors $4k + 1$, and $2p$ itself is even. For primes, Proposition 2 settles the observation recorded in A096467 [6]. Both propositions follow from classical facts and may well be known.

Limitations

The computation covers longest sides up to 2000; the rows for 5000 and 10,000 in Table 2, the range of $H(N)/N^{1.5}$ and the checks of Propositions 2 and 3 up to 10,000 use the published OEIS table instead of our own count. The growth exponent near 1.5 is measured on a finite range and is not proved. The classes in Q3 depend on the definition of gluing (integer pieces, foot of the altitude inside the side); other definitions, for example counting differences of triangles as decompositions, would merge some classes, and both versions are reported. Seventy-six zeros of $a(n)$ up to 2000 are not explained by the propositions. Finally, the numbers depend on a computer program; this risk is reduced by two enumerations with different parametrisations and by seven comparisons with eleven OEIS sequences, but it is not removed completely.

Conclusion

There are 40,764 Heronian triangles with longest side at most 2000, and the count agrees with every OEIS table it was compared with. My forecast of about 300 triangles up to 100 was close to the true value of 321. The number of triangles grows faster than the longest side, roughly like $N^{1.5}$ on the studied range, but $a(n) = 0$ for 333 values of $n \leq 2000$. A zero is impossible when n has a prime factor $4k + 1$ and certain when n is a power of a prime $4k + 3$ or twice such a prime greater than 3. Right triangles form only 4.9% of the list. About a third of the triangles are glued from two integer right triangles, a third are differences of two such triangles, and a third have no integer altitude; the last group grows with n . In every case the altitude onto the longest side splits the triangle into two right triangles with rational sides, and the denominators of every altitude and of the pieces it cuts off contain only primes $4k + 1$.

Author contributions and use of AI tools. The author chose sequence A054875 for the study from three options proposed by his mentor, who recommended it; sent the first description of the problem and the observation about the altitude onto the longest side (in Russian, 26 September 2026); sent the list of all Heronian triangles with longest side at most 10 with a proof (4 and 8 October 2026), which he worked out himself and wrote up with the help of Gemini (Google); wrote the three research questions from examples suggested by the mentor (6 October) and made the forecast (9 October); and found the papers [1, 2, 3, 8]. The mentor chose the range $n \leq 2000$ and, working with a generative AI tool (Claude, Anthropic), wrote the programs, designed the four classes, made the figures and tables, formulated and proved Lemmas 2 and 3 and Propositions 1, 2, 3 and 4, translated the author's Russian text and drafted the English text. The proofs were checked in an independent review by a separate AI agent. Every number in the paper comes from the output of the programs, and every cited work was opened and checked, except the article by Cheney mentioned in the introduction, which is not cited.

Data availability. The OEIS tables are publicly available at <https://oeis.org>. The complete list of 40,764 triangles with their classes and the programs that produce every number in this paper are available from the author's mentor on request.

Conflict of interest. The author declares no conflict of interest.

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