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UNIFORMLY ANALYTIC CONVEX SPACES

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Abstract: Let X be a complex quasi-Banach space and $2 \le q < \infty$. We prove that X is analytically q-uniformly convexifiable (i.e. admits an equivalent uniformly analytic convex quasi-norm of power type q) if and only if, for every symmetric quasi-Banach function space E on $(0, \infty)$ with the Fatou property whose dilation operators satisfy $\|D_a\| \to 0$ as $a \to 0+$, the q-th power of the q-square function $S^{(q)}(F)$ (respectively, of the conditional q-square function $\sigma^{(q)}(F)$) of every X-valued Hardy martingale F is dominated in the norm of E by the q-th power of the corresponding sharp maximal function. The proof combines a stopping-time good-λ estimate for Hardy martingales with the transfer lemma of Alday and Dauitbek, which converts good-λ inequalities into norm inequalities in symmetric spaces. The result extends the known characterization in Orlicz spaces with a moderate Young function and yields new characterizations in Lebesgue spaces L_r $(0 < r < \infty)$ and Lorentz spaces.

Keywords: uniformly analytic convex space, analytic q-uniform convexity, Hardy martingale, q-square function, sharp maximal function, symmetric quasi-Banach function space, good-λ inequality, Orlicz space, Lorentz space

1. Introduction

The interplay between the geometry of a Banach (or quasi-Banach) space X and the behaviour of X-valued martingales is one of the central themes of modern functional analysis. In the real setting, the classical results of Pisier [12] show that a Banach space admits an equivalent q-uniformly convex norm if and only if the q-square function of every X-valued martingale is dominated by the martingale itself in L_p. In harmonic analysis, however, one usually works with complex spaces, and the relevant geometry is then of an *analytic* nature. Globevnik [7] introduced complex uniform convexity, Davis, Garling and Tomczak-Jaegermann [6] studied *PL*-convexity of quasi-normed spaces, Bukhvalov and Danilevich [14] defined the analytic Radon–Nikodým property, Edgar [13] connected it with the convergence of analytic martingales, and Xu [4, 5] introduced uniform H-convexity and proved renorming theorems through inequalities for Hardy martingales.

Following Liu and Bekjan [3] and the doctoral dissertation [2], a complex quasi-Banach space is called *uniformly analytic convex* (analytically uniformly convex) if the modulus $h_q^X(\varepsilon)$ defined in (1) below is positive, and *analytically q-uniformly convex* if $h_q^X(\varepsilon) \ge c\,\varepsilon^q$. This property is strictly weaker than ordinary uniform convexity: for instance, $L_1[0,1]$ is analytically uniformly

convex, although it is not uniformly convex and even fails the Radon–Nikodým property [16]. In [2, 3] it was shown that analytic q-uniform convexifiability of X is characterized by weak type, strong (p,p) and Φ-inequalities for the q-square function of X-valued Hardy martingales. In particular, Theorem 1.4.1 of [2] states that, for a moderate Young function Φ, the space X is analytically q-uniformly convexifiable if and only if the q-square function $S^{(q)}(F)$ (respectively, the conditional q-square function $\sigma^{(q)}(F)$) is controlled in the Orlicz space L_Φ by the sharp maximal function $F_q^\#$ (respectively, $\widetilde{F}_q^\#$). The proof relies on a stopping-time good-λ inequality and on a classical distributional lemma for Young functions.

Recently, Alday and Dauitbek [1] characterized analytic q-uniform convexifiability by means of norm inequalities $\|S^{(q)}(F)\|_E \le C\|F^{¿}\|_E$ valid in *every* symmetric quasi-Banach function space E on $(0,\infty)$. The key tool of [1] is Lemma 3.2, which transfers an arbitrary good-λ inequality into a norm inequality in a symmetric space E by means of the dilation operators D_a.

The purpose of the present paper is to combine these two approaches. We replace the Orlicz norm $\|\cdot\|_\Phi$ in Theorem 1.4.1 of [2] by the norm of an arbitrary symmetric quasi-Banach function space E and prove that the resulting inequalities still characterize uniformly analytic convex spaces (Theorem 1 below). The Orlicz-space result of [2] is recovered as a special case (Corollary 1), and new examples are obtained for Lebesgue and Lorentz spaces (Corollary 2).

2. Preliminaries

2.1. Analytic uniform convexity. Throughout, X is a complex quasi-Banach space with quasi-norm $\|\cdot\|$ and quasi-triangle constant $\rho \ge 1$, i.e. $\|x+y\| \le \rho(\|x\|+\|y\|)$ for $x, y \in X$. Let T be the unit circle with normalized Lebesgue measure $dm = d\theta/2\pi$. For $0 < p < \infty$ denote by $\widetilde{H}_p(X)$ the closure in $L_p(T;X)$ of the analytic trigonometric polynomials $P(\theta) = \sum_{k=0}^{N} x_k e^{ik\theta}$, $x_k \in X$, and by $\hat{f}(0)$ the zeroth Fourier coefficient of $f \in \widetilde{H}_p(X)$. For $\varepsilon > 0$ put

$$h_p^X(\varepsilon) = \inf\left\{\left|\|f\|_p - 1\right| : \|\hat{f}(0)\| = 1,\ \|f - \hat{f}(0)\|_p \ge \varepsilon,\ f \in \widetilde{H}_p(X)\right\}. \tag{1}$$

Definition 1 [2, 3]. The space X is called *uniformly analytic convex* if $h_p^X(\varepsilon) > 0$ for all $\varepsilon > 0$ and some $0 < p < \infty$. For $2 \le q < \infty$ the space X is called *analytically q-uniformly convex* if there is $c > 0$ such that $h_q^X(\varepsilon) \ge c\varepsilon^q$ for $0 \le \varepsilon \le 1$, and *analytically q-uniformly convexifiable* if X admits an equivalent quasi-norm under which it is analytically q-uniformly convex.

It is known that C is analytically 2-uniformly convex, that analytic q-uniform convexity forces $q \ge 2$, and that analytic q-uniform convexity implies analytic q_1-uniform convexity for $q_1 \ge q$ [2, 4].

2.2. Hardy martingales and their maximal operators. Let $\Omega = T^N$ be equipped with the product probability measure P, let B_n be the σ-algebra generated by the first n coordinates $(\theta_1, \dots, \theta_n)$, $B_0 = B_{-1} = \{\varnothing, \Omega\}$, and let $E(\cdot \mid B_n)$ denote conditional expectation. An X-valued martingale $F = (F_n)_{n \ge 0}$ is called a *Hardy martingale* if $F_0 = x \in X$ and

$$dF_n = F_n - F_{n-1} = \sum_{k=1}^{\infty} \varphi_{n,k}(\theta_1, \dots, \theta_{n-1}) e^{ik\theta_n},\ n \ge 1, \tag{2}$$

with X-valued B_{n-1}-measurable coefficients $\varphi_{n,k}$. The class of all such martingales is denoted by H. We put $F_{-1} = 0$, $dF_0 = F_0$, $\|F\|_p = ¿_n \|F_n\|_{L_p(\Omega;X)}$, and call F *q-integrable* if $\|F\|_q < \infty$. For $2 \le q < \infty$ we use the q-square function and the conditional q-square function

$$S_n^{(q)}(F) = \left(\sum_{k=0}^{n} \|dF_k\|^q\right)^{1/q},\quad \sigma_n^{(q)}(F) = \left(\sum_{k=0}^{n} E\left(\|dF_k\|^q \mid B_{k-1}\right)\right)^{1/q}, \tag{3}$$

$S^{(q)}(F) = ¿_n S_n^{(q)}(F)$, $\sigma^{(q)}(F) = ¿_n \sigma_n^{(q)}(F)$, $S_{-1}^{(q)}(F) = \sigma_{-1}^{(q)}(F) = 0$. Following Section 1.4 of [2], we use the sharp maximal operators

$$F_{nq}^\# = ¿_{m \ge n}\left(E\left(\|F_m - F_{n-1}\|^q \mid B_n\right)\right)^{1/q},\quad \widetilde{F}_{nq}^\# = ¿_{m \ge n}\left(E\left(\|F_m - F_n\|^q \mid B_n\right)\right)^{1/q}, \tag{4}$$

and $F_q^{\#}=¿_{n\ge 0}F_{nq}^{\#}$, $\widetilde{F}_q^{\#}=¿_{n\ge -1}\widetilde{F}_{nq}^{\#}$. Note that $F_{nq}^{\#}$ and $\widetilde{F}_{nq}^{\#}$ are B_n-measurable, and that $\left(F_{0q}^{\#}\right)^q=\left(\widetilde{F}_{-1,q}^{\#}\right)^q=¿_m E\|F_m\|^q$, because $B_0=B_{-1}$ is trivial and $F_{-1}=0$. Taking $m=n$ in (4) we obtain the pointwise estimates

$$\|dF_n\|^q\le\left(F_{nq}^{\#}\right)^q, E\left(\|dF_n\|^q\,|\,B_{n-1}\right)\le\left(\widetilde{F}_{n-1,q}^{\#}\right)^q, n\ge 0. \tag{5}$$

We shall use the following two results of [2].

Lemma 1 [2, Lemma 1.3.1]. Let $2\le q<\infty$. If X is analytically q-uniformly convexifiable, then there is a constant $C>0$ such that for every q-integrable $F\in H$

$$E\left(\sum_{j=n}^{m}\|dF_j\|^q\,|\,B_n\right)\le C\,E\left(\|F_m-F_{n-1}\|^q\,|\,B_n\right), 0\le n\le m, \tag{6}$$

$$E\left(\sum_{j=n+1}^{m}\|dF_j\|^q\,|\,B_n\right)\le C\,E\left(\|F_m-F_n\|^q\,|\,B_n\right), -1\le n\le m. \tag{7}$$

(For $n=-1$ inequality (7) coincides with (6) for $n=0$, because $B_{-1}=B_0$.)

Lemma 2 [2, Corollary 1.2.3]. Let $q\ge 2$. Then X is analytically q-uniformly convexifiable if and only if $S^{(q)}(F)<\infty$ a.e. for every $F\in H$ with $\|F\|_\infty<\infty$.

2.3. Symmetric function spaces. For a measurable function f on a σ-finite measure space let $d(\lambda;f)=\mu\{|f|>\lambda\}$ be its distribution function and $f^{¿}(t)=inf\{\lambda>0: d(\lambda;f)\le t\}$, $t>0$, its decreasing rearrangement. Recall that $d\left(f^{¿}(s);f\right)\le s$ for every $s>0$. A quasi-Banach function space E on $(0,\infty)$ is called *symmetric* if $f\in E$, $g^{¿}\le f^{¿}$ imply $g\in E$ and $\|g\|_E\le\|f\|_E$; in particular $\|f\|_E=\|f^{¿}\|_E$. We denote by $K_E\ge 1$ the quasi-triangle constant of E. For a random variable ξ on (Ω,P) we set $\|\xi\|_E=\|\xi^{¿}\|_E$, where $\xi^{¿}$ (vanishing on ¿) is the rearrangement of ξ with respect to P. For $a>0$ the dilation operator is $(D_af)(t)=f(t/a)$, $t>0$; one has $(D_af)^{¿}=D_af^{¿}$.

Following [1], we denote by S the class of symmetric quasi-Banach function spaces E on $(0,\infty)$ such that

(E1) E has the *Fatou property*: if $0\le f_k\uparrow f$ a.e., then $\|f\|_E=\lim ¿_k\|f_k\|_E¿$ (both sides may be infinite);

(E2) every D_a is bounded on E and

$$\lim ¿_{a\to 0^{+¿}}\|D_a\|_{E\to E}=0.¿¿ \tag{8}$$

Condition (8) is exactly the decay property of the dilation operators used in the proof of Lemma 3.2 of [1]; it holds whenever the lower Boyd index p_E of E is positive (see [1, 9, 10, 11, 15]).

Examples. (a) $E=L_p(0,\infty)$, $0<p<\infty$: $\|D_a\|_{E\to E}=a^{1/p}$.

(b) Lorentz spaces $E=L_{p,r}(0,\infty)$, $0<p<\infty$, $0<r\le\infty$, with $\|f\|_{p,r}=\left(\int_0^\infty\left(t^{1/p}f^{¿}(t)\right)^r dt/t\right)^{1/r}$: since $(D_af)^{¿}=D_af^{¿}$, the substitution $t=as$ gives $\|D_a\|_{E\to E}=a^{1/p}$.

(c) Orlicz spaces. Let Φ be a moderate Young function, $\Phi(2t)\le C_\Phi\Phi(t)$ [2], and let $L_\Phi(0,\infty)$ carry the Luxemburg norm $\|f\|_\Phi=inf\{s>0:\int_0^\infty\Phi(|f(t)|/s)\,dt\le 1\}$. By convexity $C_\Phi\ge 2$. Given $0<a<1$, choose the least $k\in N$ with $aC_\Phi^k\le 1$. For every $s>\|f\|_\Phi$,

$$\int_0^\infty\Phi\left(\frac{2^k|f(t/a)|}{s}\right)dt=a\int_0^\infty\Phi\left(\frac{2^k|f(u)|}{s}\right)du\le aC_\Phi^k\int_0^\infty\Phi\left(\frac{|f(u)|}{s}\right)du\le 1,$$

hence $\|D_af\|_\Phi\le 2^{-k}\|f\|_\Phi$. Since $k\ge\log_2(1/a)/\log_2 C_\Phi$, we get $\|D_a\|_{L_\Phi\to L_\Phi}\le a^{1/\log_2 C_\Phi}\to 0$ as $a\to 0^{+¿¿}$. Thus $L_\Phi\in S$. For a random variable ξ this norm coincides with the norm $inf\{s>0: E\Phi(\xi/s)\le 1\}$ used in [2].

All spaces in (a)–(c) have the Fatou property, so they belong to S.

3. From good-λ inequalities to symmetric norms

Definition 2 [1, Definition 3.1; 2, Definition 0.0.3]. A pair (ξ,η) of nonnegative random variables satisfies a *good-λ inequality* if there exist $\alpha>1$ and, for every $\beta\in(0,1)$, constants $\varepsilon(\beta)>0$, $\delta(\beta)>0$ with $\varepsilon(\beta)\to 0$ as $\beta\to 0^{+}$, such that

$$P(\xi>\alpha\lambda)\le\varepsilon(\beta)P(\xi>\lambda)+\delta(\beta)P(\eta>\beta\lambda),\ \lambda>0. \tag{9}$$

The following lemma is Lemma 3.2 of [1]; we give its proof in the form needed below, keeping track of the constants and of the finiteness assumption required to absorb the first term.

Lemma 3. Let $E\in S$ and let (ξ,η) satisfy (9). Choose $\beta_0\in(0,1)$ such that $\alpha K_E\|D_{2\varepsilon(\beta_0)}\|_{E\to E}\le\frac{1}{2}$. If $\|\xi\|_E<\infty$, then

$$\|\xi\|_E\le\frac{2\alpha K_E\|D_{2\delta(\beta_0)}\|_{E\to E}}{\beta_0}\|\eta\|_E. \tag{10}$$

Proof. The choice of β_0 is possible by (8), because $\varepsilon(\beta)\to 0$. Write $\varepsilon=\varepsilon(\beta_0)$, $\delta=\delta(\beta_0)$. Fix $t>0$ and put

$$\lambda_1=\xi^{*}\left(\frac{t}{2\varepsilon}\right),\ \lambda_2=\frac{1}{\beta_0}\eta^{*}\left(\frac{t}{2\delta}\right),\ \lambda=max\{\lambda_1,\lambda_2\}.$$

Since $\lambda\ge\lambda_1$ and $\beta_0\lambda\ge\eta^{*}(t/2\delta)$, the property $d(f^{*}(s);f)\le s$ and (9) give

$$d(\alpha\lambda;\xi)\le\varepsilon\,d(\lambda_1;\xi)+\delta\,d(\eta^{*}(t/2\delta);\eta)\le\varepsilon\cdot\frac{t}{2\varepsilon}+\delta\cdot\frac{t}{2\delta}=t.$$

By the definition of the rearrangement, $\xi^{*}(t)\le\alpha\lambda\le\alpha\,\lambda_1+\alpha\,\lambda_2$, i.e.

$$\xi^{*}\le\alpha\,D_{2\varepsilon}\xi^{*}+\frac{\alpha}{\beta_0}D_{2\delta}\eta^{*}\ \text{on}\ (0,\infty). \tag{11}$$

Since E is symmetric (hence an ideal lattice) with quasi-triangle constant K_E, (11) yields

$$\|\xi\|_E\le\alpha K_E\|D_{2\varepsilon}\|_{E\to E}\|\xi\|_E+\frac{\alpha K_E}{\beta_0}\|D_{2\delta}\|_{E\to E}\|\eta\|_E\le\frac{1}{2}\|\xi\|_E+\frac{\alpha K_E}{\beta_0}\|D_{2\delta}\|_{E\to E}\|\eta\|_E.$$

As $\|\xi\|_E<\infty$, the first term can be absorbed into the left-hand side, which gives (10). ▫

4. Main result

The next proposition is the stopping-time part of the proof of Theorem 1.4.1 in [2]; we present it in a form that does not depend on the Orlicz norm.

Proposition 1. Let $2\le q<\infty$, let X be analytically q-uniformly convexifiable and let C be the constant of Lemma 1. Then for every q-integrable $F\in H$ and all $\lambda>0$, $\beta>0$

$$P\left(S^{(q)}(F)^q>2\lambda\right)\le C\beta\,P\left(S^{(q)}(F)^q>\lambda\right)+P\left(\left(F_q^{\#}\right)^q>\beta\lambda\right), \tag{12}$$

$$P\left(\sigma^{(q)}(F)^q>2\lambda\right)\le C\beta\,P\left(\sigma^{(q)}(F)^q>\lambda\right)+P\left(\left(\widetilde{F}_q^{\#}\right)^q>\beta\lambda\right). \tag{13}$$

In particular, the pairs $\left(S^{(q)}(F)^q,\left(F_q^{\#}\right)^q\right)$ and $\left(\sigma^{(q)}(F)^q,\left(\widetilde{F}_q^{\#}\right)^q\right)$ satisfy the good-λ inequality (9) with $\alpha=2$, $\varepsilon(\beta)=C\beta$ and $\delta(\beta)=1$.

Proof. (12). Write $S_n=S_n^{(q)}(F)$ and define the stopping times (with $inf\,\varnothing=\infty$)

$$\tau=inf\{n\ge 0:S_n^q>2\lambda\},\ \nu=inf\{n\ge 0:S_n^q>\lambda\},\ \mu=inf\{n\ge 0:\left(F_{nq}^{\#}\right)^q>\beta\lambda\}.$$

Then $\nu\le\tau$, $\{\tau<\infty\}=\{S^{(q)}(F)^q>2\lambda\}$, $\{\nu<\infty\}=\{S^{(q)}(F)^q>\lambda\}$ and $\{\mu<\infty\}=\{\left(F_q^{\#}\right)^q>\beta\lambda\}$. Hence

$$P(\tau<\infty)\le P(\tau<\infty,\nu<\mu)+P(\mu<\infty). \tag{14}$$

On $\{\tau<\infty\}$ we have $S_{\nu-1}^q\le\lambda$ and $S_\tau^q>2\lambda$, so $\sum_{j=\nu}^{\infty}\|dF_j\|^q\ge S_\tau^q-S_{\nu-1}^q>\lambda$. The set $\{\nu=n<\mu\}=\{\nu=n\}\cap\{\mu>n\}$ belongs to B_n, and on it $\left(F_{nq}^{\#}\right)^q\le\beta\lambda$. Letting $m\to\infty$ in (6) (monotone convergence for conditional expectations) we get $E\left(\sum_{j\ge n}\|dF_j\|^q\,|\,B_n\right)\le C\left(F_{nq}^{\#}\right)^q$. Therefore, by Chebyshev's inequality,

$$P(\tau<\infty,\nu<\mu)\le\frac{1}{\lambda}\sum_{n\ge0}E\left[1_{\{\nu=n<\mu\}}E\left(\sum_{j\ge n}\|dF_j\|^q\,|\,B_n\right)\right]\le\frac{C}{\lambda}\sum_{n\ge0}E\left[1_{\{\nu=n<\mu\}}\left(F^{\#}_{nq}\right)^q\right]\le C\beta\,P(\nu<\infty).$$

Together with (14) this proves (12).

(13). Write $\sigma_n=\sigma_n^{(q)}(F)$; note that σ_{n+1} is B_n-measurable. Define the stopping times with respect to $(B_n)_{n\ge-1}$

$$\tau=inf\{n\ge-1:\sigma_{n+1}^q>2\lambda\},\theta=inf\{n\ge-1:\sigma_{n+1}^q>\lambda\},\mu=inf\{n\ge-1:\left(\widetilde{F}^{\#}_{nq}\right)^q>\beta\lambda\}.$$

Then $\theta\le\tau$, $\{\tau<\infty\}=\{\sigma^{(q)}(F)^q>2\lambda\}$, $\{\theta<\infty\}=\{\sigma^{(q)}(F)^q>\lambda\}$, $\{\mu<\infty\}=\{\left(\widetilde{F}^{\#}_{q}\right)^q>\beta\lambda\}$. On $\{\tau<\infty\}$ we have $\sigma_\theta^q\le\lambda$, hence $\sum_{k\ge\theta+1}E\left(\|dF_k\|^q\,|\,B_{k-1}\right)\ge\sigma_{\tau+1}^q-\sigma_\theta^q>\lambda$. Since $\{\theta=n<\mu\}\in B_n\subset B_{k-1}$ for $k\ge n+1$, the tower property and (7) (with $m\to\infty$) give

$$E\left[1_{\{\theta=n<\mu\}}\sum_{k\ge n+1}E\left(\|dF_k\|^q\,|\,B_{k-1}\right)\right]=E\left[1_{\{\theta=n<\mu\}}E\left(\sum_{k\ge n+1}\|dF_k\|^q\,|\,B_n\right)\right]\le C\beta\lambda\,P(\theta=n<\mu).$$

Summing over $n\ge-1$ and applying Chebyshev's inequality as above, we get $P(\tau<\infty,\theta<\mu)\le C\beta P(\theta<\infty)$, and (13) follows. ▫

Now we can state the main result. It is Theorem 1.4.1 of [2] with the Orlicz norm $\|\cdot\|_\Phi$ replaced by the norm of an arbitrary space $E\in S$, as in Lemma 3.2 of [1].

Theorem 1. Let X be a complex quasi-Banach space and $2\le q<\infty$. The following statements are equivalent:

(i) X is analytically q-uniformly convexifiable;

(ii) for every $E\in S$ there is a constant $C_E>0$ such that

$$\left\|S^{(q)}(F)^q\right\|_E\le C_E\left\|\left(F^{\#}_q\right)^q\right\|_E,\ \forall F\in H;\qquad(15)$$

(iii) for every $E\in S$ there is a constant $C_E>0$ such that

$$\left\|\sigma^{(q)}(F)^q\right\|_E\le C_E\left\|\left(\widetilde{F}^{\#}_q\right)^q\right\|_E,\ \forall F\in H;\qquad(16)$$

(iv) inequality (15) holds for some $E=L_p(0,\infty)$, $0<p<\infty$;

(v) inequality (16) holds for $E=L_1(0,\infty)$.

Proof. (i) ⇒ (ii). Fix $E\in S$ and $F\in H$. We may assume $\eta:=\left(F^{\#}_q\right)^q$ satisfies $\|\eta\|_E<\infty$. Then $\eta<\infty$ a.e., and since $\left(F^{\#}_{0q}\right)^q=¿_m E\|F_m\|^q$ is a constant not exceeding η, the martingale F is q-integrable. For $N\in N$ consider the stopped martingale $F^N=(F_{n\wedge N})_{n\ge0}$; it belongs to H, is q-integrable, $S^{(q)}(F^N)=S_N^{(q)}(F)$, and directly from (4)

$$\left(F^N\right)^{\#}_{nq}\le F^{\#}_{nq}(n\le N),\left(F^N\right)^{\#}_{nq}=0(n>N),\text{so}\left(F^N\right)^{\#}_{q}\le F^{\#}_{q}.$$

Put $\xi_N=S_N^{(q)}(F)^q$. By Proposition 1 applied to F^N,

$$P(\xi_N>2\lambda)\le C\beta\,P(\xi_N>\lambda)+P(\eta>\beta\lambda),\lambda,\beta>0,$$

so (ξ_N,η) satisfies (9) with $\alpha=2$, $\varepsilon(\beta)=C\beta$, $\delta(\beta)=1$. Moreover, by (5), $\xi_N=\sum_{k=0}^N\|dF_k\|^q\le(N+1)\eta$, whence $\|\xi_N\|_E\le(N+1)\|\eta\|_E<\infty$. Choose β_0 with $2K_E\|D_{2C\beta_0}\|_{E\to E}\le\frac{1}{2}$ (possible by (8)). Lemma 3 gives

$$\left\|S_N^{(q)}(F)^q\right\|_E\le\frac{4K_E\|D_2\|_{E\to E}}{\beta_0}\left\|\left(F^{\#}_q\right)^q\right\|_E,\qquad(17)$$

with a constant independent of N and F. Since $S_N^{(q)}(F)^q\uparrow S^{(q)}(F)^q$, the Fatou property (E1) yields (15) with $C_E=4K_E\|D_2\|_{E\to E}/\beta_0$.

(i) ⇒ (iii). The argument is the same, with (13) instead of (12). For the stopped martingale, $\sigma^{(q)}(F^N)=\sigma_N^{(q)}(F)$ and $\left(\widetilde{F^N}\right)^{\#}_q\le\widetilde{F}^{\#}_q$, while (5) gives $\sigma_N^{(q)}(F)^q=\sum_{k=0}^N E\left(\|dF_k\|^q\,|\,B_{k-1}\right)\le(N+1)\left(\widetilde{F}^{\#}_q\right)^q$, so that Lemma 3 is applicable. Again q-integrability follows from $\left(\widetilde{F}^{\#}_{-1,q}\right)^q=¿_m E\|F_m\|^q$.

(ii) ⇒ (iv) and (iii) ⇒ (v) are trivial, since $L_p(0,\infty)\in S$ by Example (a).

(iv) ⇒ (i). Let $F\in H$ with $\|F\|_\infty<\infty$. Then $\|F_m-F_{n-1}\|\le 2\rho\|F\|_\infty$, so $F_q^{\#}\le 2\rho\|F\|_\infty$ and

$$\left\|S^{(q)}(F)^q\right\|_{L_p}\le C_E(2\rho\|F\|_\infty)^q<\infty.$$

Hence $S^{(q)}(F)<\infty$ a.e., and X is analytically q-uniformly convexifiable by Lemma 2.

(v) ⇒ (i). Let $\|F\|_\infty<\infty$. By the tower property and monotone convergence, $ES^{(q)}(F)^q=E\sigma^{(q)}(F)^q$, and $\widetilde{F}_q^{\#}\le 2\rho\|F\|_\infty$. By (16) with $E=L_1$,

$$ES^{(q)}(F)^q=\left\|\sigma^{(q)}(F)^q\right\|_{L_1}\le C_{L_1}(2\rho\|F\|_\infty)^q<\infty,$$

so $S^{(q)}(F)<\infty$ a.e., and Lemma 2 yields (i). ▫

Corollary 1 [2, Theorem 1.4.1]. Let $2\le q<\infty$ and let Φ be a moderate Young function. If X is analytically q-uniformly convexifiable, then there is $C>0$ such that for all $F\in H$

$$\left\|S^{(q)}(F)^q\right\|_\Phi^{1/q}\le C\left\|\left(F_q^{\#}\right)^q\right\|_\Phi^{1/q},\left\|\sigma^{(q)}(F)^q\right\|_\Phi^{1/q}\le C\left\|\left(\widetilde{F}_q^{\#}\right)^q\right\|_\Phi^{1/q}.$$

Conversely, each of these inequalities implies that X is analytically q-uniformly convexifiable.

Proof. By Example (c), $L_\Phi\in S$, so the direct part follows from Theorem 1 by taking q-th roots. For the converse, if $\|F\|_\infty<\infty$, then $\|\left(F_q^{\#}\right)^q\|_\Phi<\infty$ and hence $S^{(q)}(F)<\infty$ a.e. (respectively, $\|\sigma^{(q)}(F)^q\|_\Phi<\infty$; then $E\Phi\left(\sigma^{(q)}(F)^q/s\right)\le 1$ for some $s>0$, Jensen's inequality gives $E\sigma^{(q)}(F)^q<\infty$, and $ES^{(q)}(F)^q=E\sigma^{(q)}(F)^q<\infty$). Lemma 2 completes the proof. ▫

Corollary 2. Let $2\le q<\infty$. The following are equivalent:

(i) X is analytically q-uniformly convexifiable;

(ii) for every $0<r<\infty$ there is $C_r>0$ such that $\|S^{(q)}(F)\|_{L_r(\Omega)}\le C_r\|F_q^{\#}\|_{L_r(\Omega)}$ for all $F\in H$;

(iii) for all $0<p<\infty$, $0<s\le\infty$ there is $C_{p,s}>0$ such that $\|S^{(q)}(F)^q\|_{L_{p,s}}\le C_{p,s}\|\left(F_q^{\#}\right)^q\|_{L_{p,s}}$ for all $F\in H$.

Proof. Apply Theorem 1 with $E=L_{r/q}$ (note that $\|\xi^q\|_{L_{r/q}}=\|\xi\|_{L_r}^q$) and with $E=L_{p,s}$, which belong to S by Examples (a), (b). ▫

Remark. Theorem 1 complements Theorem 3.5 of [1], where analytic q-uniform convexifiability is characterized by the inequality $\|S^{(q)}(F)\|_E\le C\|F^{¿}\|_E$ involving the ordinary maximal function $F^{¿}=¿_n\|F_n\|$. Both results are obtained from a good-λ inequality through Lemma 3.2 of [1]; the difference lies in the controlling function. The sharp functions $F_q^{\#}$ and $\widetilde{F}_q^{\#}$ measure the conditional oscillation of the martingale rather than its size. In this sense inequalities (15)–(16) are analytic counterparts of the distribution function inequalities of Burkholder [8] for real-valued martingales.

5. Conclusion

We have shown that the class of uniformly analytic convex spaces, more precisely the class of analytically q-uniformly convexifiable complex quasi-Banach spaces, is characterized by inequalities between the q-square functions $S^{(q)}(F)$, $\sigma^{(q)}(F)$ of X-valued Hardy martingales and their sharp maximal functions $F_q^{\#}$, $\widetilde{F}_q^{\#}$ in an arbitrary symmetric quasi-Banach function space E whose dilation operators satisfy $\|D_a\|_{E\to E}\to 0$ as $a\to 0^{+¿¿}$. The proof combines the stopping-time good-λ estimate from the doctoral dissertation [2] with the transfer principle of Alday and Dauitbek [1, Lemma 3.2]. The result contains the Orlicz-space characterization of [2] as a special case and yields new characterizations in Lebesgue spaces L_r for all $0<r<\infty$ and in Lorentz spaces $L_{p,s}$. A natural next step is to obtain similar symmetric-norm characterizations for the variation operators $W_q^q(F)$, $\widetilde{W}_q^q(F)$ and for the martingale transforms considered in Section 1.4 of [2], and for q-uniform PL-convexity via analytic martingales.

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